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FIXED POINT THEOREMS IN BANACH SPACE AND 2-BANACH SPACE

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ABSTRACT

We generalize the result of Goebel and Zlotkiewiez [5] and also we prove fixed point theorems in Banach and 2-Banach spaces in this paper.

Subject Classification (AMS 2000) : 47H10, 54H25

Keywords- *Banach Space, 2-Banach Space, Closed and Convex set, Coincidence point.*

I. INTRODUCTION

Let X be a Banach class space and C be a closed subset of X. The well known Banach contraction principle state a contraction mapping of C into itself has unique fixed point. The same result holds if we assume that only some powers of mapping are contraction but it is not true for non-expansive mappings. Some fixed point theorem have been studied by many mathematicians say Browder and Petryshyn [3],Belluce and Kirk [2] , Diaz and Mateaf [4] and many others for the existence of fixed points of non-expansive maps defined on a closed , bounded and convex subset of a uniformly convex Banach space and in a space with a normed structure.

Afif Ben Amar, [8] has introduce Fixed point theorems for the sum of (ws)-compact and asymptotically Φ-nonexpansive mappings .

Whether these results can be extended to mappings with a non-expansive iteration in general is not true. However, Goebel and Zlotkiewiez [5] have given an idea that these problem with the some and proved the following restriction.

Theorem 1. Let F be a mapping of a Banach space X into itself. If F satisfies condition

(i) $F^2 = I$

(ii) $\|Fx - Fy\| \leq \alpha \|x - y\|$

for every x,y $\in$ X, where $0 \leq \alpha < 2$

then F has atleast one fixed point.

Iseki [6] and Achari [1] obtained a further generalization of Goebel-Zlotkiewiez [5].
In a paper Khan and Imdad [7] extended the result due to Goebel and Zlotkiewiez [5] for mapping satisfying more general condition. He has also proved some coincidence theorem and obtained similar results in 2-Banach spaces.

II. MAIN RESULT

In this section ,an attempt is made to show results due to Goebel and Zlotkiewiez [5] can be extended for mapping satisfying more general conditions. We apply our results to prove some Fixed and Coincidence points in Banach and 2-Banach spaces.

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Theorem 2. Let X be a Banach space and C be a closed and convex subset of X. Let E:C $\rightarrow$C satisfy the condition

(i)$E^2 = I$

(ii) $\|Ex - Ey\| \leq \alpha \frac{[\|x - Ex\| + \|y - Ey\|]^2}{\|x - Ex\| + \|y - Ex\|} + \beta\|x - y\|$

for every x,y $\in$C where α , β are non –negative and $0 \leq \frac{8}{3}\alpha + \frac{1}{2}\beta < 1$. Then E has atleast one fixed point.

Proof Let x be an arbitrary point of C and H= $\frac{1}{2}$(I+E). Put y=Hx, z=Ey, u=2y-z . Then we have

$$\|z - x\| = \|Ey - x\| = \|Ey - E^2 x\|$$
$$\leq \alpha \frac{[\|y - Ey\| + \|Ex - E^2 x\|]^2}{\|y - Ey\| + \|Ex - E^2 y\|} + \beta\|y - Ex\|$$
$$\leq \alpha \frac{[\|Ex - x\| + \|Ex - x\|]^2}{\|Ex - x\| + \frac{1}{2}\|Ex - x\|} + \frac{\beta}{2}\|Ex - x\|$$
$$\leq \frac{8}{3}\alpha\|Ex - x\| + \frac{\beta}{2}\|Ex - x\|$$
$$\leq \left(\frac{16\alpha + 3\beta}{6}\right)\|Ex - x\|$$

and

$$\|u - x\| = \|2y - z - x\| = \|Ex - Ey\|$$
$$\leq \alpha \frac{[\|x - Ex\| + \|y - Ey\|]^2}{\|x - Ex\| + \|y - Ex\|} + \beta\|x - y\|$$
$$\leq \alpha \frac{[\|Ex - x\| + \|Ex - x\|]^2}{\|Ex - x\| + \frac{1}{2}\|Ex - x\|} + \frac{\beta}{2}\|Ex - x\|$$
$$\leq \left(\frac{16\alpha + 3\beta}{6}\right)\|Ex - x\|$$

Hence

$$\|z - u\| \leq \|z - x\| + \|x - u\|$$
$$\leq \|z - x\| + \|u - x\|$$
$$\leq \left(\frac{16\alpha + 3\beta}{6}\right)\|Ex - x\| + \left(\frac{16\alpha + 3\beta}{6}\right)\|Ex - x\|$$

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$$\leq\left(\frac{16\alpha+3\beta}{3}\right)\|Ex-x\|.$$

On the other hand we have

$$\|H^2x-Hx\|=\|Hy-y\|$$
$$=\left\|\frac{1}{2}(I+E)y-y\right\|$$
$$=\frac{1}{2}\|y-Ey\|$$
$$=\frac{1}{2}\|x-Ex\|$$
$$=\|Hx-x\|.$$

The sequence $\{x_n\}$ defined by $x_n=H^nx$ is a Cauchy sequence in X and since X is a complete , so that H^nx converges to some element $x_0\in X$, i.e., $\lim_{n\to\infty}x_n=x_0$.

Now

$$\|x_0-Hx_0\|\leq\|x_0-Hx_n\|+\|Hx_n-Hx_0\|$$
$$\leq\|x_0-Hx_n\|+\left\|\frac{1}{2}(I+E)x_n-\frac{1}{2}(I+E)x_0\right\|$$
$$\leq\|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\|Ex_n-Ex_0\|$$
$$\leq\|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\left[\alpha\frac{\{\|x_n-Ex_n\|+\|x_0-Ex_0\|\}^2}{\|x_n-Ex_n\|+\|x_0-Ex_n\|}+\beta\|x_n-x_0\|\right]$$
$$\leq\|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\left[\alpha\frac{\{\|x_n-(2Hx_n-x_n)\|+\|x_0-(2Hx_0-x_0)\|\}^2}{\|x_n-(2Hx_n-x_n)\|+\|x_0-(2Hx_n-x_n)\|}+\beta\|x_n-x_0\|\right]$$
$$\leq\|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\left[\alpha\frac{\{\|2x_n-2Hx_n\|+\|2x_0-2Hx_0\|\}^2}{\|2x_n-2Hx_n\|+\|x_0+x_n-2Hx_n\|}+\beta\|x_n-x_0\|\right]$$

Taking $\lim_{n\to\infty}x_n=x_0$.

$$\leq\|x_0-Hx_0\|+2\alpha\|x_0-Hx_0\|$$
$$\leq(1+2\alpha)\|x_0-Hx_0\|.$$

We have $x_0=Hx_0$, hence $x_0=Ex_0$ i.e., x_0 is a fixed point of E.

If

$$\|y-Ey\|=\|Hx-E(Hx)\|$$

$$= \left\| \frac{1}{2}(I+E)x - E\frac{1}{2}(I+E)x \right\|$$

$$= \frac{1}{2}\left\| x - E^2 x \right\|$$

$$\le \frac{1}{2}\left[\|x - Ex\| + \|Ex - E^2 x\| \right]$$

$$\le \frac{1}{2}\left[\|x - Ex\| + \|Ex - x\| \right]$$

$$\le \|x - Ex\|.$$

Thus

$$\|y - Ey\| \le \|x - Ex\|$$

Now

$$\|H^2 x - Hx\| = \|Hy - y\|$$

$$= \left\| \frac{1}{2}(y + Ey) - y \right\|$$

$$= \frac{1}{2}\|y - Ey\|$$

$$\le \frac{1}{2}\|x - Ex\|$$

$$\le \|Hx - x\|$$

I.e., $\|H^2 x - Hx\| \le \|Hx - x\|$.

We claim that $H^n x$ is Cauchy sequence in X and by completeness of X , $H^n x$ converges to some point $x^* \in X$ i.e., $\lim_{n\to\infty} H^n x = x^*$, which implies that $Hx^* = x^*$.

Hence $Ex^* = x^*$ is a fixed point of E.

Theorem 3. Let E be a mapping of 2-Banach space X into itself such that the following hold:

(i)$E^2 = I$

(ii) $\|Ex - Ey, a\| \le \alpha \dfrac{\left[\|x - Ex, a\| + \|y - Ey, a\|\right]^2}{\|x - Ex, a\| + \|y - Ex, a\|} + \beta \|x - y, a\|$

for every x,y,a $\in$ X where α , β are non –negative and $0 \le \frac{8}{3}\alpha + \frac{1}{2}\beta < 1$.If $\dim X \ge 2$ then E has at least one fixed point.

Proof Let x be an arbitrary point of X and H= $\frac{1}{2}$ (I+E). Put y=Hx, z=Ey, u=2y-z . Then we have

$$\|z-x,a\| = \|Ey-x,a\| = \|Ey-E^2x,a\|$$

$$\le \alpha \frac{\left[\|y-Ey,a\|+\|Ex-E^2x,a\|\right]^2}{\|y-Ey,a\|+\|Ex-E^2y,a\|} + \beta\|y-Ex,a\|$$

$$\le \alpha \frac{\left[\|Ex-x,a\|+\|Ex-x,a\|\right]^2}{\|Ex-x,a\|+\frac{1}{2}\|Ex-x,a\|} + \frac{\beta}{2}\|Ex-x,a\|$$

$$\le \frac{8}{3}\alpha\|Ex-x,a\| + \frac{\beta}{2}\|Ex-x,a\|$$

$$\le \left(\frac{16\alpha+3\beta}{6}\right)\|Ex-x,a\|$$

and

$$\|u-x,a\| = \|2y-z-x,a\| = \|Ex-Ey,a\|$$

$$\le \alpha \frac{\left[\|x-Ex,a\|+\|y-Ey,a\|\right]^2}{\|x-Ex,a\|+\|y-Ex,a\|} + \beta\|x-y,a\|$$

$$\le \alpha \frac{\left[\|Ex-x,a\|+\|Ex-x,a\|\right]^2}{\|Ex-x,a\|+\frac{1}{2}\|Ex-x,a\|} + \frac{\beta}{2}\|Ex-x,a\|$$

$$\le \left(\frac{16\alpha+3\beta}{6}\right)\|Ex-x,a\|$$

Hence

$$\|z-u,a\| \le \|z-x,a\| + \|x-u,a\|$$

$$\le \|z-x,a\| + \|u-x,a\|$$

$$\le \left(\frac{16\alpha+3\beta}{6}\right)\|Ex-x,a\| + \left(\frac{16\alpha+3\beta}{6}\right)\|Ex-x,a\|$$

$$\le \left(\frac{16\alpha+3\beta}{3}\right)\|Ex-x,a\|.$$

On the other hand we have

$$\|H^2x-Hx,a\| = \|Hy-y,a\|$$

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$$= \left\| \frac{1}{2}(I+E)y - y, a \right\|$$
$$= \frac{1}{2}\|y - Ey, a\|$$
$$= \frac{1}{2}\|x - Ex, a\|$$
$$= \|Hx - x, a\|.$$

The sequence $\{x_n\}$ defined by $x_n = H^n x$ is a Cauchy sequence in X and since X is a complete , so that $H^n x$ converges to some element $x_0 \in X$, i.e., $\lim_{n\to\infty} x_n = x_0$.

Now

$$\|x_0 - Hx_0, a\| \le \|x_0 - Hx_n, a\| + \|Hx_n - Hx_0, a\|$$
$$\le \|x_0 - Hx_n, a\| + \left\| \frac{1}{2}(I+E)x_n - \frac{1}{2}(I+E)x_0, a \right\|$$
$$\le \|x_0 - Hx_n, a\| + \frac{1}{2}\|x_n - x_0, a\| + \frac{1}{2}\|Ex_n - Ex_0, a\|$$
$$\le \|x_0 - Hx_n, a\| + \frac{1}{2}\|x_n - x_0, a\|$$
$$+ \frac{1}{2}\left[\alpha \frac{\{\|x_n - Ex_n, a\| + \|x_0 - Ex_0, a\|\}^2}{\|x_n - Ex_n, a\| + \|x_0 - Ex_n, a\|} + \beta\|x_n - x_0, a\| \right]$$
$$\le \|x_0 - Hx_n, a\| + \frac{1}{2}\|x_n - x_0, a\|$$
$$+ \frac{1}{2}\left[\alpha \frac{\{\|x_n - (2Hx_n - x_n), a\| + \|x_0 - (2Hx_0 - x_0), a\|\}^2}{\|x_n - (2Hx_n - x_n), a\| + \|x_0 - (2Hx_n - x_n), a\|} + \beta\|x_n - x_0, a\| \right]$$
$$\le \|x_0 - Hx_n, a\| + \frac{1}{2}\|x_n - x_0, a\|$$
$$+ \frac{1}{2}\left[\alpha \frac{\{\|2x_n - 2Hx_n, a\| + \|2x_0 - 2Hx_0, a\|\}^2}{\|2x_n - 2Hx_n, a\| + \|x_0 + x_n - 2Hx_n, a\|} + \beta\|x_n - x_0, a\| \right]$$

Taking $\lim_{n\to\infty} x_n = x_0$.

$$\le \|x_0 - Hx_0, a\| + 2\alpha\|x_0 - Hx_0, a\|$$
$$\le (1+2\alpha)\|x_0 - Hx_0, a\|.$$

We have $x_0 = Hx_0$, hence $x_0 = Ex_0$ i.e., x_0 is a fixed point of E.

If

$$\|y - Ey, a\| = \|Hx - E(Hx), a\|$$

$$= \left\| \frac{1}{2}(I+E)x - E\frac{1}{2}(I+E)x, a \right\|$$
$$= \frac{1}{2}\left\| x - E^2x, a \right\|$$
$$\leq \frac{1}{2}\left[\left\| x - Ex, a \right\| + \left\| Ex - E^2x, a \right\| \right]$$
$$\leq \frac{1}{2}\left[\left\| x - Ex, a \right\| + \left\| Ex - x, a \right\| \right]$$
$$\leq \left\| x - Ex, a \right\|.$$

Thus

$$\left\| y - Ey, a \right\| \leq \left\| x - Ex, a \right\|$$

Now

$$\left\| H^2x - Hx, a \right\| = \left\| Hy - y, a \right\|$$
$$= \left\| \frac{1}{2}(y + Ey) - y, a \right\|$$
$$= \frac{1}{2}\left\| y - Ey, a \right\|$$
$$\leq \frac{1}{2}\left\| x - Ex, a \right\|$$
$$\leq \left\| Hx - x, a \right\|$$

I.e., $\left\| H^2x - Hx, a \right\| \leq \left\| Hx - x, a \right\|$.

We claim that $H^n x$ is Cauchy sequence in X and by completeness of X , $H^n x$ converges to some point $x^* \in X$ i.e., $\lim_{n\to\infty} H^n x = x^*$, which implies that $Hx^* = x^*$.

Hence $Ex^* = x^*$ is a fixed point of E.

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